JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 16, No. 5, September–October 1993

Feedback Gain Sensitivities of Closed-Loop Modal Parameters of Controlled Structures

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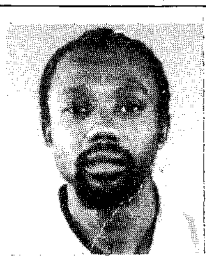
Computer algebra (MACSYMA) has been used to derive the characteristic polynomials that determine the closed-loop frequencies and damping coefficients of output-feedback controlled structures. These expressions show explicitly how the locations of the actuator/sensor pairs and the displacement and velocity feedback gains influence the frequencies and damping parameters of the closed-loop system modes. For lightly coupled modes, simple relations are obtained between the modal parameters and the coordinates of the sensor/actuator pairs as well as the displacement and velocity feedback gains. Using the example of a cantilevered uniform beam controlled by a single sensor/actuator pair, numerical results are used to illustrate the sensitivity of the closed-loop modal parameters to the placement of the sensor/actuator pair as well as the feedback gains. Such results help answer questions about optimal placement of sensor/actuator pairs for the active control of a flexible structure.

Nomenclature

a_i, a_j	= measurement coordinates for ith and jth controllers
b_i, b_j	= force application coordinates for ith and jth controllers
$C_j^{(i)}$	= symbolic coefficient of jth power of p in the i-term derivation
$G(x,\xi,t,\tau)$	= Green's function
g_i, g_j	= displacement feedback gains of ith and jth controllers
h_i, h_j	= velocity feedback gains of ith and jth controllers
I	= identity matrix
L	= number of discrete attachments (controllers); beam length
$L_{x,t}\{\ \}$	= partial differential operator
p	= Laplace variable, $\sigma + i\omega$
p_{0k}	= kth parameter of baseline system
$Q(x,t), Q(x,p)$	= system response and its Laplace transform
t	= time
$W(x,\xi,p)$	= system transfer function
$W_{ci}(p)$	= transfer function of ith controller
$w(x,t), w(x,p)$	= forcing function and its Laplace transform
x, x_1, x_2	= spatial coordinates
α_i	= coefficient of the ith power of p in characteristic polynomial
β, β_k	= modal weights
$\{\gamma\}, \gamma_j$	= vector with elements defined in Eq. (15)
$\delta(\ )$	= Dirac delta function
η	= spatial coordinate
ξ	= spatial coordinate
$\sigma, \sigma_{0n_1}, \sigma_{1n_1}$	= exponential growth rate
φ_{0k}	= kth orthonormal modal function for baseline system
τ	= time
$[\Omega], \Omega_{i,j}$	= matrix with elements defined in Eq. (12)
$\omega, \omega_{0n_1}, \omega_{1n_1}$	= frequency
$\{\omega\}, \omega_i$	= vector with elements defined in Eq. (13)

Introduction

LARGE flexible structures such as those being deployed for space-based applications usually require some form of active control technology to maintain vibrational response within tolerable levels.[1] Because spaceborne structures cannot afford the weight penalties of classical vibration control devices such as absorbers or isolators, a lot of effort has been devoted to various means of actively controlling the dynamic characteristics of these structures. Such techniques use an external source of energy to apply controlling forces (and/or moments) on the structure, which are determined in some relationship to the measured or estimated response of the structure. From a review of the literature on this subject, the approach of choice for the design of the active control schemes appears to follow the paths of modern control theory that involves optimal state-space feedback control[1-7] or output feedback control.[2,8] Preliminary to the application of optimal control techniques, a discretization of the equations of motion



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Received Aug. 9, 1991; presented as Paper 92-2322 at the AIAA/ASME/ASCE/AHS/ASC 33rd Structures, Structural Dynamics, and Materials Conference, Dallas, TX, April 13–15, 1992; revision received Oct. 1, 1992; accepted for publication Oct. 30, 1992.

is accomplished using the finite element method, the modal decomposition method, or outright lumping of the parameters of the structure. The state variables are the (generalized) displacements and (generalized) velocities of the structure. In the case of state-space feedback, the state of the system is estimated from measurements at selected coordinates. That estimate is used to derive the feedback gains using a method based on Pointryagin's principle for solving constrained optimization problems, which involves the computation of a positive-definite matrix satisfying the algebraic matrix-Riccati equation.[4] Output feedback control does not use the entire state-space estimate for feedback; instead only the measured responses are used, the advantages being that the practical implementation of the controller is simpler and errors associated with estimating of unmeasured responses are eliminated.[8]

A considerable amount of numerical computation is involved in implementing the methods currently used in practice, and it often happens that the designer is not afforded the benefit of simple results that might aid his or her intuition in the design process. Traditionally, regardless of what a computer program gives as a result, the engineer still needs some method of applying simple intuitive considerations for assessing the feasibility of the design. In classical vibration control, and even in classical control theory, such intuitive assessments are provided by studying the placement of the closed-loop system parameters—resonant frequency and damping in the case of vibration control and the placement of poles and zeroes in the case of classical control theory. Some efforts at understanding the closed-loop parameters of actively controlled structures have been reported recently.[9,10]

Techniques based on computer algebra[11,12] have been developed, which permit the derivation of the transfer functions (Laplace transform of the Green's functions) of the system resulting from the attachment of discrete dynamic substructures to a distributed parameter baseline structure.[13,14] It is assumed that the algebraic forms of the transfer function of the baseline structure as well as those of the discrete attachments are known. The mathematical form of the system transfer functions permits the direct determination of the system parameters, which are the complex values of the Laplace variable at which singularities of the transfer function occur. When the measured outputs are proposed as the feedback variables in an active controller design, the resulting system is mathematically equivalent to that of the attachment of discrete "substructures,"[15] the transfer functions of which are given by expressions involving the gain constants and the Laplace variable. In this research effort, these computer-algebraic tools have been used to derive explicit expressions for the closed-loop system parameters, in terms of the sensor/actuator placement and the displacement and velocity feedback gains. The goal is to develop insights into how the placement of the sensor/actuator pairs influences the sensitivity of the closed-loop system parameters to the feedback gains.

Analysis of Active Controller Effects

The objective of this section is to present the formulation of the equations that were used to derive the transfer function of the system resulting from the attachment of a finite number of discrete linear output feedback controllers to a distributed parameter, baseline system such as a flexible structure. These derivations follow the same lines as those previously published,[13,14] but because these references may not be easily accessible to the reader, the relevant parts of the derivation are repeated here for completeness. The dynamic responses of a distributed parameter system are solutions to partial integro-differential equations, which can be represented operationally as

$$L_{x,t}\{Q(x_2,t)\} = w(x_1,t) \tag{1}$$

where $L_{x,t}\{\ \}$ is an integro-differential operator that maps the responses $Q(x_2,t)$ onto the excitations $w(x_1,t)$ subject to appropriate boundary and initial conditions on $Q(x_2,t)$; x_1 is in the spatial domain of the excitations and x_2 is in the spatial domain of the responses. For linear operators, $G(x,\xi,t,\tau)$ is defined such that

$$L_{x,t}\{G(x_2,\xi,t,\tau)\} = \delta(x_1-\xi)\delta(t-\tau) \tag{2}$$

The response of the system can be conveniently written in terms of the Green's function as

$$Q(x_2,t) = \iint G(x_2,\xi,t,\tau)w(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \tag{3}$$

since

$$\begin{aligned} L_{x,t}\{Q(x_2,t)\} &= \iint L_{x,t}\{G(x_2,\xi,t,\tau)\}w(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\ &= \iint \delta(x_1-\xi)\delta(t-\tau)w(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\ &= w(x_1,t) \end{aligned} \tag{4}$$

Butkovskyi[16,17] has proposed the introduction of a linear distributed block—in analogy to the lumped parameter block in classical control theory—to represent the input-output relationship between $Q(x_2,t)$ and $w(x_1,t)$. Thus the schematic of Fig. 1a is equivalent to the relationship expressed in Eq. (3). For dynamical systems whose responses to stationary excitations are stationary, i.e., the Green's function is stationary in time, the analysis can be simplified by considering the Laplace transform of the equations of motion. The role of the Green's function is played by the transfer function, and the relationship of the Laplace transform of the response $Q(x_2,p)$ to that of the excitation $w(x_1,p)$ is given by

$$Q(x_2,p) = \int G(x_2,\xi,p)w(\xi,p)\,\mathrm{d}\xi \tag{5}$$

where $p=\sigma+i\omega$ is the Laplace variable, σ is the exponential growth rate, and ω is the frequency. This relationship is also depicted schematically in Fig. 1b.

Modeling of Active Controller Attachments

Implementing the active controller design with displacement and velocity feedback involves applying excitation forces at some spatial coordinate $x_1=b_i$ that are proportional to displacements and velocities measured at $x_2=a_i$. For example, the ith controller excitation force could be written as

$$w_c(b_i,t) = g_iQ(a_i,t) + h_i\dot{Q}(a_i,t) \tag{6}$$

where g_i and h_i are the displacement and velocity feedback gains of the ith controller, respectively, $i=1,2,\ldots,L$, L being the total number of controllers. The Laplace transform of Eq. (6) results in a relationship that is used to define the transfer function of the ith controller as

$$\begin{aligned} W_{ci}(p) &= \frac{w_c(b_i,p)}{Q(a_i,p)} \\ &= g_i + h_ip \end{aligned} \tag{7}$$

The schematic that represents the combined interconnected system of the baseline structure and the L active controllers is shown in Fig. 2. The transfer function of the combined system shown in Fig. 2 is given by the following integral equation[16]:

$$W(x.\xi,p) = \int W_T(x,\eta,p)W(\eta,\xi,p)\,\mathrm{d}\eta + W_0(x,\xi,p) \tag{8}$$

where

$$\begin{aligned} W_T(x,\xi,p) &= \int W_0(x,\eta,p)\sum_{i=1}^{L}\delta(\eta-b_i)W_{ci}(p)\delta(\xi-a_i)\,\mathrm{d}\eta \\ &= \sum_{i=1}^{L} W_0(x,b_i,p)W_{ci}(p)\delta(\xi-a_i) \end{aligned} \tag{9}$$

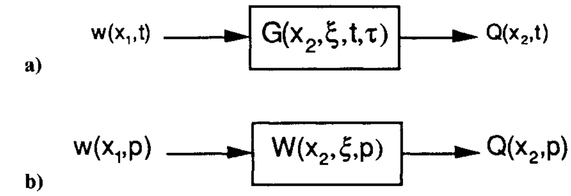

Fig. 1 Linear distributed block: a) Green's function representation and b) transfer function representation.

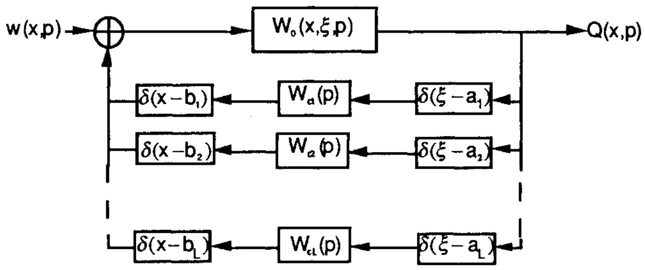

Fig. 2 Schematic of interconnection of linear feedback controllers to distributed parameter baseline structure.

Substituting Eq. (9) into Eq. (8) and performing the integration, the result is

$$W(x,\xi,p) = \sum_{i=1}^{L} W_0(x,b_i,p)W_{ci}(p)W(a_i,\xi,p) + W_0(x,\xi,p) \tag{10}$$

To solve for the quantities $W(a_i,\xi,p)$, $i=1,2,\dots,L$, both sides of Eq. (10) are successively multiplied by $\delta(x-a_m)$, and integrations are performed over the x domain for $m=1,2,\dots,L$ to get

$$W(a_m,\xi,p) = \sum_{i=1}^{L} W_0(a_m,b_i,p)W_{ci}(p)W(a_i,\xi,p) + W_0(a_m,\xi,p) \tag{11}$$

Equation (11) is a system of L linear equations defining L unknown quantities. If an $(L\times L)$ matrix $[\Omega]$ is defined such that its elements are

$$\Omega_{i,j} = W_0(a_j,b_i,p)W_{ci}(p) \tag{12}$$

and if an $(L\times 1)$ vector $\{\omega\}$ is defined such that its elements are

$$\omega_i = W(a_i,\xi,p) \tag{13}$$

then the system of Eq. (11) for $m=1,2,\dots,L$ can be written in a compact form as

$$\{\omega\} = [\Omega]\{\omega\} + \{\gamma\} \tag{14}$$

where $\{\gamma\}$ is an $(L\times 1)$ vector whose elements are

$$\gamma_j = W_0(a_j,\xi,p) \tag{15}$$

From Eq. (14)

$$\{\omega\} = [I-\Omega]^{-1}\{\gamma\} \tag{16}$$

where I is the $(L\times L)$ identity matrix.

Application of Computer Algebra

The general algebraic form of the transfer function of the baseline distributed system is taken to be[17,18]

$$W_0(x,\xi,p) = \sum_{k=1}^{\infty}\left(\frac{1}{\beta_k^2}\,\frac{\varphi_{0k}(x)\varphi_{0k}(\xi)}{p^2-p_{0k}^2}\right) \tag{17}$$

where $\varphi_{0k}(x)$ is the kth orthonormal modal function for the baseline structure, p_{0k} is the corresponding modal parameter, and β_k^2 is the weighting factor. Although the summation in Eq. (17) includes an infinite number of terms, the practical implementation of that expression requires that only a finite number of terms be retained. The ability to retain a given number of modes in the algebraic derivation depends on the power and memory of the computer as well as the number of discrete modifications to the baseline structure. The substitution of Eq. (17) into Eq. (16) and the subsequent evaluation and simplification of the transfer function of the combined system as shown in Eq. (10) is performed using the set of MACSYMA routines presented in the Appendix.

The computer-algebraic results that will be presented in the next section have been generalized to the case of an arbitrary number of discrete attachments. This was done by mathematical induction, based on the results provided by MACSYMA for different numbers of attachments specified in the function calls.

Computer-Algebraic Results

Some results of the derivation of the characteristic polynomials for an arbitrary number of attachments using one and two terms in the baseline transfer function series are presented in this section. These derivations were performed on the Symbolics 3620. A uniform one-dimensional baseline structure was assumed in this study, so that $\beta_k^2=\beta^2$ for all the ks.

One-Term Derivation

To simplify the analysis of the sensitivity of individual modal parameters to the active controller design variables such as sensor/actuator placement and displacement and velocity feedback gains, the assumption is made that the modes of the baseline structure are uncoupled (which is reasonable if the baseline structure is very lightly damped). This allows the transfer function given by Eq. (17), for frequencies close to the resonance of a specified mode, to be approximated by one term in that series, which represents the contribution of that mode. The characteristic polynomial for the closed-loop system is then obtained as

$$\text{Polynomial}(1) = C_2^{(1)}p^2 + C_0^{(1)}(p) \tag{18}$$

where

$$C_0^{(1)}(p) = -\beta^2 p_{0n_1}^2 - \sum_{i=1}^{L}\left\{W_{ci}(p)\varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i)\right\} \tag{19}$$

and

$$C_2^{(1)} = \beta^2 \tag{20}$$

Note that the polynomial in Eq. (18) is not yet fully explicit in p until the expressions for $W_{ci}(p)$ are substituted into Eq. (19). If these functions are as shown in Eq. (7), then

$$C_0^{(1)}(p) = -\beta^2 p_{0n_1}^2 - \sum_{i=1}^{L}\left\{(g_i+h_i p)\varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i)\right\} \tag{21}$$

So that for this class of active controller design, and using the one-term approximation to the baseline transfer function, the characteristic polynomial in p whose roots are the system parameters, is given by

$$\text{Polynomial}(1) = \{\beta^2\}p^2 - \left\{\sum_{i=1}^{L}\left\{h_i\varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i)\right\}\right\}p - \left\{\beta^2 p_{0n_1}^2 + \sum_{i=1}^{L}\left\{g_i\varphi_{0n_1}(a_i)\phi_{0n_1}(b_i)\right\}\right\} \tag{22}$$

Two-Term Derivation

For closely coupled modes, the preceding approximation may not give adequate results. The next level of refinement is to consider the combined contribution of two neighboring modes. For this case, two terms are retained in Eq. (17), and the polynomial derived by the computer-algebraic routine is

$$\text{Polynomial}(2) = C_4^{(2)}p^4 + C_2^{(2)}(p)p^2 + C_0^{(2)}(p) \tag{23}$$

where

$$C_4^{(2)} = (\beta^2)^2 \tag{24}$$

$$C_2^{(2)}(p) = -(\beta^2)^2 \sum_{j=1}^{2} \left\{p_{0n_j}^2\right\} - \sum_{i=1}^{L} \left\{W_{ci}(p) \sum_{j=1}^{2} \left[\beta^2 \varphi_{0n_j}(a_i)\varphi_{0n_j}(b_i)\right]\right\}$$

$$C_2^{(2)}(p) = -(\beta^2)^2 \sum_{j=1}^{2} \left\{p_{0n_j}^2\right\} - \sum_{i=1}^{L} \left\{W_{ci}(p)\left[\beta\varphi_{0n_1}(a_i) \;\; \beta\varphi_{0n_2}(a_i)\right]\begin{bmatrix}\beta\varphi_{0n_1}(b_i)\\ \beta\varphi_{0n_2}(b_i)\end{bmatrix}\right\} \tag{25}$$

and

$$C_0^{(2)}(p) = \left(\beta^2 p_{0n_1}^2\right)\cdot\left(\beta^2 p_{0n_2}^2\right) + \sum_{i=1}^{L}\left\{W_{ci}(p)\left[\beta\varphi_{0n_1}(a_i) \;\; \beta\varphi_{0n_2}(a_i)\right]\begin{bmatrix}p_{0n_2}^2 & 0\\ 0 & p_{0n_1}^2\end{bmatrix} \times \begin{Bmatrix}\beta\varphi_{0n_1}(b_i)\\ \beta\varphi_{0n_2}(b_i)\end{Bmatrix}\right\} + \sum_{i=1}^{L}\sum_{j>i}^{L} \times \left\{\begin{matrix} W_{ci}(p)W_{cj}(p)\left(\left[\varphi_{0n_1}(a_i) \;\; -\varphi_{0n_2}(a_i)\right]\begin{Bmatrix}\varphi_{0n_2}(a_j)\\ \varphi_{0n_1}(a_j)\end{Bmatrix}\right) \\ \times\left(\left[\varphi_{0n_1}(b_i) \;\; -\varphi_{0n_2}(b_i)\right]\begin{Bmatrix}\varphi_{0n_2}(b_j)\\ \varphi_{0n_1}(b_j)\end{Bmatrix}\right)\end{matrix}\right\} \tag{26}$$

As mentioned earlier, the explicit dependence of the polynomial on the Laplace variable p will be determined when the appropriate expressions are substituted for the functions $W_{ci}(p)$. If the controller transfer functions given in Eq. (7) are substituted into Eqs. (25) and (26), then following a collection of the coefficients of the powers of p in the polynomial of Eq. (23), the result is

$$\text{Polynomial}(2) = \alpha_4 p^4 + \alpha_3 p^3 + \alpha_2 p^2 + \alpha_1 p + \alpha_0 \tag{27}$$

where

$$\alpha_4 = (\beta^2)^2 \tag{28}$$

$$\alpha_3 = -\sum_{i=1}^{L}\left\{h_i\left[\beta\varphi_{0n_1}(a_i) \;\; \beta\varphi_{0n_2}(a_i)\right]\begin{bmatrix}\beta\varphi_{0n_1}(b_i)\\ \beta\varphi_{0n_2}(b_i)\end{bmatrix}\right\} \tag{29}$$

$$\alpha_2 = -(\beta^2)^2\sum_{j=1}^{2}\left\{p_{0n_j}^2\right\} - \sum_{i=1}^{L} \times \left\{g_i\left[\beta\varphi_{0n_1}(a_i) \;\; \beta\varphi_{0n_2}(a_i)\right]\begin{bmatrix}\beta\varphi_{0n_1}(b_i)\\ \beta\varphi_{0n_2}(b_i)\end{bmatrix}\right\} + \sum_{i=1}^{L}\sum_{j>i}^{L} \times \left\{\begin{matrix} h_i h_j\left(\left[\varphi_{0n_1}(a_i) \;\; -\varphi_{0n_2}(a_i)\right]\begin{Bmatrix}\varphi_{0n_2}(a_j)\\ \varphi_{0n_1}(a_j)\end{Bmatrix}\right) \\ \times\left(\left[\varphi_{0n_1}(b_i) \;\; -\varphi_{0n_2}(b_i)\right]\begin{Bmatrix}\varphi_{0n_2}(b_j)\\ \varphi_{0n_1}(b_j)\end{Bmatrix}\right)\end{matrix}\right\} \tag{30}$$

$$\alpha_1 = \sum_{i=1}^{L}\left\{h_i\left[\beta\varphi_{0n_1}(a_i) \;\; \beta\varphi_{0n_2}(a_i)\right]\begin{bmatrix}p_{0n_2}^2 & 0\\ 0 & p_{0n_1}^2\end{bmatrix}\begin{bmatrix}\beta\varphi_{0n_1}(b_i)\\ \beta\varphi_{0n_2}(b_i)\end{bmatrix}\right\} + \sum_{i=1}^{L}\sum_{j>i}^{L}\left\{\begin{matrix}(g_i h_j + g_j h_i)\times\left(\left[\varphi_{0n_1}(a_i) \;\; -\varphi_{0n_2}(a_i)\right]\begin{bmatrix}\varphi_{0n_2}(a_j)\\ \varphi_{0n_1}(a_j)\end{bmatrix}\right) \\ \times\left(\left[\varphi_{0n_1}(b_i) \;\; -\varphi_{0n_2}(b_i)\right]\begin{bmatrix}\varphi_{0n_2}(b_j)\\ \varphi_{0n_1}(b_j)\end{bmatrix}\right)\end{matrix}\right\} \tag{31}$$

and

$$\alpha_0 = \left(\beta^2 p_{0n_1}^2\right)\cdot\left(\beta^2 p_{0n_2}^2\right) + \sum_{i=1}^{L}\left\{g_i\left[\beta\varphi_{0n_1}(a_i) \;\; \beta\varphi_{0n_2}(a_i)\right]\begin{bmatrix}p_{0n_2}^2 & 0\\ 0 & p_{0n_1}^2\end{bmatrix}\begin{bmatrix}\beta\varphi_{0n_1}(b_i)\\ \beta\varphi_{0n_2}(b_i)\end{bmatrix}\right\} + \sum_{i=1}^{L}\sum_{j>i}^{L}\left\{\begin{matrix}(g_i g_j)\times\left(\left[\varphi_{0n_1}(a_i) \;\; -\varphi_{0n_2}(a_i)\right]\begin{bmatrix}\varphi_{0n_2}(a_j)\\ \varphi_{0n_1}(a_j)\end{bmatrix}\right) \\ \times\left(\left[\varphi_{0n_1}(b_i) \;\; -\varphi_{0n_2}(b_i)\right]\begin{bmatrix}\varphi_{0n_2}(b_j)\\ \varphi_{0n_1}(b_j)\end{bmatrix}\right)\end{matrix}\right\} \tag{32}$$

As far as the computer-algebraic procedure is concerned, any number of terms in baseline transfer function expressions can be used. However, it is clear that such expressions will be complicated and are better left in the memory of the computer.

Sensitivity Analysis

Lightly Coupled Modes

The purpose of this paper is to examine simple but intuitive relationships between the closed-loop system parameters and the design variables of the active controller such as the actuator/sensor pair placement and the displacement and velocity feedback gains. It is assumed that the uncontrolled structure is very lightly damped, and hence its orthonormal modes are weakly coupled. This assumption allows the one-term derivation that resulted in Eq. (22). Moreover, since interest here lies in sensitivities of the modal parameters rather than their exact values, this simple result remains useful for developing insight into how the controller design affects the closed-loop parameters. Equating the polynomial of Eq. (22) to zero and solving for the real and imaginary parts of p_{1n_1}, with the assumption that the baseline structure is undamped (i.e., $p_{0n_1} = i\omega_{0n_1}$), the frequencies and exponential growth rates of the closed-loop modes are

$$\omega_{1n_1} = \sqrt{\omega_{0n_1}^2 - \left\{\frac{1}{2\beta^2}\sum_{i=1}^{L}\left[h_i\varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i)\right]\right\}^2 - \frac{1}{\beta^2}\sum_{i=1}^{L}\left[g_i\varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i)\right]} \tag{33}$$

$$\sigma_{1n_1} = \frac{1}{2\beta^2} \sum_{i=1}^{L} \left[h_i \varphi_{0n_1}(a_i) \varphi_{0n_1}(b_i) \right] \tag{34}$$

As suspected, the damping of the closed-loop system is controlled by the velocity feedback gain, whereas the frequency is affected by both the velocity and displacement feedback gains. The sensor/actuator placement controls how these feedback gains influence the frequencies and damping of the participating modes of the baseline structure. This effect is more clearly displayed by the sensitivity of the exponential growth rate to the velocity feedback gains as well as the sensitivity of the frequencies to the displacement feedback gains

$$\frac{\partial \sigma_{1n_1}}{\partial h_i} = -\frac{1}{2} \frac{\partial \omega_{1n_1}^2}{\partial g_i} = \frac{1}{2\beta^2} \varphi_{0n_1}(a_i) \phi_{0n_1}(b_i) \tag{35}$$

When the sensor and actuator of the ith controller are collocated (i.e., $a_i = b_i$), the right-hand side (RHS) of Eq. (35) is positive, and all structural modes with nonzero values of the orthonormal modes at the controller location are guaranteed to be damped by a negative velocity feedback gain. For controllers with noncollocated sensor/actuator pairs, the RHS of Eq. (35) could be positive or negative for different modes of the baseline structure. The baseline modes for which the product $\varphi(a_i)\varphi(b_i)$ is positive will be damped for negative velocity feedback gains, whereas those modes with a negative product of $\varphi(a_i)\varphi(b_i)$ could experience destabilization as a result of negative velocity feedback gain.

$$\frac{\partial \sigma_{1n_1}}{\partial g_i} = 0 \tag{36}$$

$$\frac{\partial \omega_{1n_1}^2}{\partial h_i} = -\left[\frac{1}{2\beta^4} \sum_{k=1}^{L} \left(h_k \varphi_{0n_1}(a_k) \varphi_{0n_1}(b_k) \right) \right] \varphi_{0n_1}(a_i) \varphi_{0n_1}(b_i) \tag{37}$$

As expected, the exponential growth rate is not affected by the displacement feedback. However, the closed-loop frequency is sensitive to the velocity feedback as well as the displacement feedback. For a single sensor/actuator pair ($L = 1$), the sign of the RHS of Eq. (37) is dictated by the sign of the velocity feedback. In the following section, the numerical example of a uniform cantilevered Euler-Bernoulli beam is used to illustrate the feedback gain sensitivities of the closed-loop growth rate, as the sensor/actuator pair placement is varied, for both collocated and noncollocated controllers.

Numerical Example

Consider a uniform cantilevered Euler-Bernoulli beam, with parameters as shown in Fig. 3, where the first five orthonormal modes have also been displayed. Suppose a single sensor/actuator pair is to be employed to control the vibrations of this beam using displacement and velocity feedback. Three situations will be examined: 1) the sensor and actuator are collocated at the same spanwise coordinate; 2) the sensor and actuator are not collocated, with the sensor always at the tip of the beam; and 3) the sensor and actuator are not collocated, with

n_1	ω_{0n_1}	A	B	C	D	$\varphi_{0n_1}(X)$
1	3.516	0.02417	4.138	0.01875	3.038	$A(B(\cos(Cx)-\cosh(Cx))-D(\sin(Cx)-\sinh(Cx)))$
2	22.034	0.00186	53.645	0.04694	54.636	
3	61.697	7.67e-5	1289.986	0.07854	1288.985	
4	120.902	0.10235	-	0.10996	-	$A(\cos(Cx)-\sin(Cx)-e^{-Cx})$
5	199.860	0.10182	-	0.14137	-	

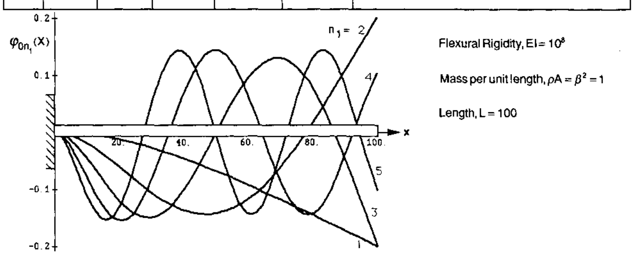

Fig. 3 The first five orthonormal modes of a uniform cantilevered Euler-Bernoulli beam.

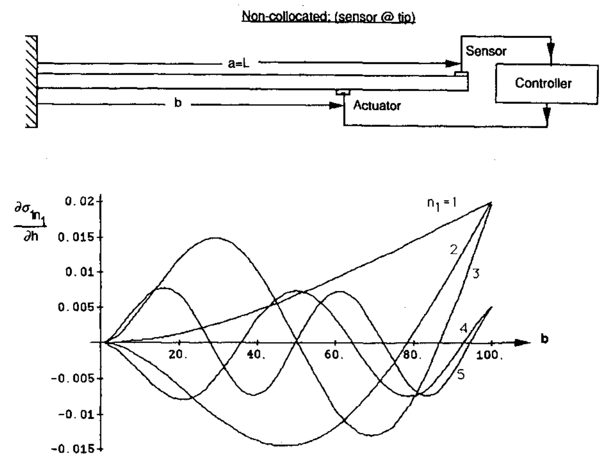

Fig. 5 Damping sensitivity to velocity feedback: noncollocated sensor/actuator pair, sensor at tip.

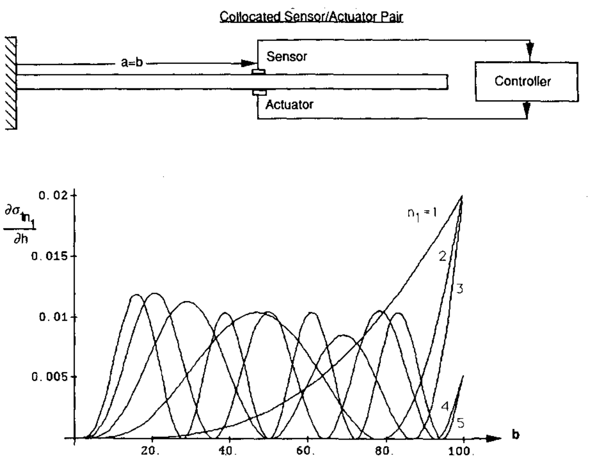

Fig. 4 Damping sensitivity to velocity feedback: collocated sensor/actuator pair.

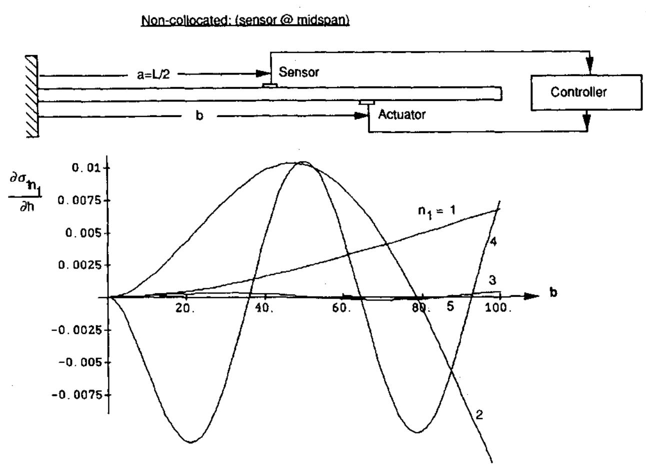

Fig. 6 Damping sensitivity to velocity feedback: noncollocated sensor/actuator pair, sensor at midspan.

the sensor at midspan. In each of these situations, the growth rate sensitivity to velocity feedback gain for different locations of the actuator will be studied. The growth rate sensitivity to velocity feedback gain is characterized by the partial derivative of the growth rate with respect to the velocity feedback gain. As given by Eq. (35), this quantity depends on the locations of the sensor and actuator. For vibration suppression it is required that the growth rate be negative, and also that its sensitivity to velocity feedback gain be of the same sign for all the modes of the structure. If the sign of this sensitivity is not the same for all modes, velocity feedback gains that suppress the vibration of certain modes will tend to cause increased response of others.

The plots of the growth rate sensitivity to velocity feedback gains for different locations of the actuator are shown in Figs. 4–6. Figure 4 is for the case where the sensor and actuator pair are collocated, whereas in Figs. 5 and 6 the sensor and actuator are not collocated. In Fig. 5 the sensor is always at the tip, and Fig. 6 is for the case where the sensor is always at midspan. When the sensor is collocated with the actuator, as in Fig. 4 for all locations of the actuator, or in Fig. 5 when the actuator is at the tip, or in Fig. 6 when the actuator is at midspan, the growth rate sensitivity to velocity feedback retains the same sign for all the modes of the beam. Collocated sensor/actuator pairs near the tip of the cantilevered beam have higher sensitivities of the growth rate of the lower modes to velocity feedback than the higher modes. This situation is reversed near the root of the beam. When the sensor is not collocated with the actuator, there are sign reversals of the growth rate sensitivity to velocity feedback for different modes of the beam. In these cases, the same velocity feedback gain that increases the damping of certain modes will reduce the damping of others. These effects are well known and are not surprising. What is interesting in the present approach is that these sensitivities can be quantified explicitly. Tradeoffs can be made with regard to the selection of modes that need maximum sensitivity at the expense of other modes that may not be critical in a given application.

This example has been presented as a simple illustration of how the results of this analysis could be used in guiding the selection of sensor/actuator placements. In most practical applications the orthonormal modes may not be available in analytical form. It may become necessary to conduct experimental modal testing to obtain the modal data required by this approach. Regardless of how the modal data is obtained, the simple expressions derived in this paper can be used to obtain preliminary assessments of the most advantageous placement of sensor/actuator pairs for active vibration control.

Conclusion

The sensitivities of closed-loop system parameters to displacement and velocity feedback gains are important considerations for the placement of sensors and actuators for active suppression of structural vibrations. Whereas optimal controller design methods using modern control theory can yield the optimal feedback gains, they do not provide insights into the placement of the sensors and the actuators for the best effect. Using computer algebra, explicit expressions have been derived for the sensitivities of the closed-loop system parameters such as resonant frequency and exponential growth rates to velocity and displacement feedback gains. These expressions make it possible to select locations for sensors and actuators at which the closed-loop system parameters have the desired sensitivities to the velocity and displacement feedback gains. Numerical examples based on a cantilevered uniform beam were used to illustrate the application of this method and to show that it is possible to conduct a rational assessment of the most advantageous locations of sensors and actuators for active control.

Appendix: MACSYMA Routines for the Derivation of Transfer Functions of the Combined System

```
W0(EXX,XXSI,PEE): = BLOCK(
    PURPOSE:"EXPRESSION FOR TRANSFER FUNCTION FOR BASELINE
    STRUCTURE - I.E. THE FUNCTION W0(X,XSI,P)"
    RAT(SUM('PHI(EXX,K)*'PHI(XXSI,K)/(PEE^2-('P0[K]^2),K,N1,N2))/'BSQ)$
GAMMA_VECTOR(XXXSI,ARGP): = BLOCK(
    GAMMA:ZEROMATRIX(NS,1),
    FOR J THRU NS
    DO SETELMX(W0('A[J],XXSI,ARGP),J,1,GAMMA))$
OMEGA_MATRIX(ARGP): = BLOCK(
    CAP_OMEGA:ZEROMATRIX(NS,NS),
    FOR I THRU NS
    DO (FOR J THRU NS
    DO (W0IJ:W0('A[J],'B[I],ARGP),
    SETELMX(W0IJ*WCP[I],I,J,CAP_OMEGA))))$
W(EXX,XXSI,PEE,N11,N22,N33): = BLOCK(
    SCALARMATRIXP:FALSE,N1:N11,N2:N22,NS:N33,
    FOR KK FROM N1 THRU N2
    DO STARTP(KK),
    FOR N THRU NS
    DO WCP[N]:RAT(SUBST(PEE,P,WC[N])),
    OMEGA_MATRIX(PEE), GAMMA_VECTOR(XXSI,PEE),
    MATRIX:IDENT(NS)-CAP_OMEGA,OMEGA:ZEROMATRIX(NS,1),
    INVERSE_MATRIX:RAT(ADJOINT(MATRIX))/RAT(DETERMINANT(MATRIX)),
    POLY:DENOM(INVERSE_MATRIX[1,1]),
    OMEGA:RAT(INVERSE_MATRIX . GAMMA),
    W1:RAT(SUM(W0(EXX,B[I],PEE)*WCP[I]*OMEGA[I,1],I,1,NS)+
    W0(EXX,XXSI,PEE)), "DONE")$
```

W1 gives the expression for the transfer function of the combined system; POLY is the characteristic polynomial of the combined system. To cast the resultant transfer function into the form of Eq. (17) for the combined system, the system parameters p_{1k} are determined as the roots of the characteristic polynomial of the system. In the previous routines, it is possible to consider any range of terms [N11,N22] in the baseline transfer function series as well as any number [N33] of discrete attachments to the baseline system.

Acknowledgments

This research was sponsored by the U.S. Air Force Office of Scientific Research. The U.S. Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation herein.

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